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The volume measure for flat connections as limit of the Yang–Mills measure

Ambar N. Sengupta

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

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Abstract

We prove that integration over the moduli space of flat connections can be obtained as a limit of integration with respect to the Yang–Mills measure defined in terms of the heat-kernel for the gauge group. In doing this we also give a rigorous proof of Witten's formula for the symplectic volume of the moduli space of flat connections. Our proof uses an elementary identity connecting determinants of matrices along with a careful accounting of certain dense subsets of full measure in the moduli space.

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1. Introduction

1.1. Summary and brief background

We work with a closed, oriented surface Σ of genus $g \geq 2$, and a compact, connected, semisimple Lie group G equipped with a bi-invariant metric. The space $\mathcal{A}$ of all connections on a principal G-bundle over Σ has a natural symplectic structure which is preserved by the pullback action $\omega \mapsto \phi^*\omega$ of the group $\mathcal{G}$ of bundle automorphisms ϕ. The moment map turns out to be $J : \omega \mapsto \Omega^\omega$, where Ω^ω denotes the curvature of any connection ω. In this setting, the Marsden–Weinstein procedure can be carried out rigorously [19] and produces a symplectic structure $\bar{\Omega}$ on the smooth strata of $J^{-1}(0)/\mathcal{G}$. Since $J^{-1}(0)$ is the set of connections with zero curvature, $J^{-1}(0)/\mathcal{G}$ is the *moduli space of flat connections*. This

E-mail address: sengupta@math.lsu.edu (A.N. Sengupta).

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space, along with the symplectic structure $\bar{\Omega}$ on it, is of interest from many different points of view (as attested to by the collection [14]). To be precise, $J^{-1}(0)/\mathcal{G}$ is not, in general, a smooth manifold but there is a subset $\mathcal{M}_g^0$ (arising from points of $J^{-1}(0)$ of "minimal" isotropy) which is a manifold and $\bar{\Omega}$ is a symplectic structure on $\mathcal{M}_g^0$.

In this paper we:

- give a rigorous proof of Witten's formula [24, formula (4.72)]

$$\mathrm{vol}_{\bar{\Omega}}(\mathcal{M}_g^0) = |Z(G)|\mathrm{vol}(G)^{2g-2} \sum_{\alpha} \frac{1}{(\dim \alpha)^{2g-2}}, \tag{1}$$

  for the *symplectic volume of the moduli space* $\mathcal{M}_g^0$ *of flat connections*, for a compact, semisimple gauge group G, over a closed oriented surface of genus $g \geq 2$ (terminology, notation, and hypotheses are explained in detail later in this introduction; note also that $\mathcal{M}_g^0$ is actually a subset of the full moduli space of flat connections).
- prove Forman's theorem [8, Theorem 1] that Wilson loop expectations in the quantum Yang–Mills theory converges to the corresponding symplectic integrals.

We will keep things as self-contained as reasonably possible and no knowledge of the moduli space of flat connections is actually necessary to understand the technical content of this paper. Indeed we shall work with a standard realization of $\mathcal{M}_g^0$ as a *finite-dimensional* manifold. Our proof has two main ingredients:

(i) a determinant identity (Proposition 1);
(ii) careful accounting of certain dense subsets of full measure in the moduli space $\mathcal{M}_g^0$ where nice properties hold.

Witten [24,25] determined the symplectic volume of the moduli space of flat connections in several different ways. One way involves the limit of the partition function of the quantum Yang–Mills theory over the surface. It is this approach, involving the heat-kernel on the structure (gauge) group, that we follow here. Forman used this approach and Witten's volume formula to prove the convergence of the Wilson loop expectations. Liu [15,16] used Forman's approach along with other ideas to study the symplectic volume and related integrals. We refer to the collection [23], and the bibliography therein, for other works concerning the symplectics of the moduli space of flat connections.

In the present paper we restrict our attention to the moduli space of flat connections without distinguishing between bundles of different topological type. The methods used here should extend to bundles of specified topology and also to the case of surfaces with boundary but this is not carried out here.

The limiting result we prove can be reformulated to give the limit of the discrete Yang–Mills measure for cell-complexes but we do not describe how this is done and deal only with the case where the surface of genus g is obtained by appropriate pasting of one-cell on the boundary of a single two-cells. (The method is described in the proof of [18, Lemma 8.5].)

We use, in several places, the existence of appropriate dense subsets. We give either proofs or exact references to proofs, when we state or use such density results. It is widespread practice in the literature on this subject to state or use without clear justification results concerning certain subsets of the moduli space of flat connections which are assumed to be dense and, implicitly, of full measure. But *much of the technical difficulty in proving the volume formula lies in taking proper account of such subsets* (which need also to be of full measure) and so we have strived to be careful about this issue. (I am thankful to the anonymous referee for stressing the necessity of having sets of full measure.)

1.2. Statement of results

We work with a compact, connected, semisimple Lie group G, whose Lie algebra LG is equipped with an Ad-invariant inner-product. The *heat-kernel* on G is a function $Q_t(x)$, for $t > 0$ and $x \in G$, satisfying the heat equation $\partial Q_t(x)/\partial t = (1/2)\Delta_G Q_t(x)$, where Δ_G is the Laplacian on G, and the initial condition $\lim_{t\downarrow 0}\int_G f(x)Q_t(x)\,\mathrm{d}x = f(e)$ for every continuous function f on G, where e is the identity in G and $\mathrm{d}x$ the Haar measure on G of unit total mass $\int_G \mathrm{d}x = 1$.

For any integer $g \geq 1$, let $K_g : G^{2g} \to G$ be the product commutator map given by

$$K_g : G^{2g} \to G : (a_1, b_1, \dots, a_g, b_g) \mapsto b_g^{-1}a_g^{-1}b_g a_g \cdots b_1^{-1}a_1^{-1}b_1 a_1. \tag{2}$$

The subset $K_g^{-1}(e)$, where e is the identity in G, of G^{2g} will be of special interest to us. The group G acts by conjugation on G^{2g}. If $A \subset G^{2g}$ is preserved by this action, denote by A^0 the set of all points on A where the isotropy is $Z(G)$, the center of G. The quotient

$$\mathcal{M}_g = \frac{K_g^{-1}(e)}{G}, \tag{3}$$

is identifiable in a standard way with the moduli space of flat G-connections over a closed, connected, oriented two-dimensional manifold of genus g, but we shall not need any detail of this (see (A.13) in Appendix A). The subset

$$\mathcal{M}_g^0 = \frac{K_g^{-1}(e)^0}{G} \tag{4}$$

(when non-empty) has a manifold structure and on $\mathcal{M}_g^0$ there is a natural symplectic form $\bar{\Omega}$. Let $\mathrm{vol}_{\bar{\Omega}}$ be the volume form corresponding to this symplectic structure; i.e. $\mathrm{vol}_{\bar{\Omega}} = (1/d!)\bar{\Omega}^{d/2}$, where $d = \dim \mathcal{M}_g^0$.

Our main result is the following theorem.

Theorem 1. *Suppose $g \geq 2$. Let f be a continuous G-conjugation-invariant function on G^{2g}, and $\tilde{f}$ the function induced on $\mathcal{M}_g^0 = K_g^{-1}(e)^0/G$. Then*

$$\lim_{t\downarrow 0}\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \frac{\mathrm{vol}(G)^{2-2g}}{|Z(G)|}\int_{\mathcal{M}_g^0}\tilde{f}\,\mathrm{d}\,\mathrm{vol}_{\bar{\Omega}}, \tag{5}$$

where the integration on the left is with respect to unit-mass Haar measure, the integration on the right is with respect to the symplectic volume measure, $|Z(G)|$ is the number of elements in the center $Z(G)$ of G, and $\mathrm{vol}(G)$ is the volume of G with respect to the Riemannian structure on G given by the Ad-invariant metric on LG.

The integral on the left in (5) arises from integration with respect to the Yang–Mills measure in the Euclidean quantum field theory of the Yang–Mills field on a compact oriented surface of genus g. We shall not need this, but a rapid account is given in Appendix A; for more details see [17] or the review [23].

Setting $f = 1$ leads, after some computation (detailed in (42)) to Witten's formula [24, formula (4.72)] for the symplectic volume of the moduli space $\mathcal{M}_g^0$:

$$\mathrm{vol}_{\bar{\Omega}}(\mathcal{M}_g^0) = |Z(G)|\mathrm{vol}(G)^{2g-2} \sum_{\alpha} \frac{1}{(\dim \alpha)^{2g-2}}, \tag{6}$$

where α runs over all irreducible representations of G.

In essence, Eq. (5) for $f = 1$ is one of the approaches used by Witten [24] to determine the volume of the moduli space.

For general f, Theorem 1 was proved by Forman [8] *using* Witten's volume formula (in fact, this is also what we shall do, but we shall also prove the volume formula (6)). For $G = SU(2)$ and $SO(3)$, the result was proved in [21].

What we shall prove in this paper is actually the limit formula:

$$\lim_{t\downarrow 0} \int_{G^{2g}} f(x) Q_t(K_g(x))\, \mathrm{d}x = \mathrm{vol}(G)^{1-2g} \int_{K_g^{-1}(e)^0} \frac{f(x)}{|\mathrm{d}K_g(x)^*|} \mathrm{d}\,\mathrm{vol}(x), \tag{7}$$

for any continuous function f on G^{2g}, where the linear map $\mathrm{d}K_g(x)^* : LG \to (LG)^{2g}$ is the adjoint of the derivative $(LG)^{2g} \to LG : H \mapsto K_g(x)^{-1}K_g'(x)(xH)$, and $\mathrm{d}\,\mathrm{vol}$ is Riemannian volume measure on the submanifold $K_g^{-1}(e)^0 \subset G^{2g}$. The known result (34) then implies (5).

The *main difficulty* in proving (7) *lies in taking proper care of the critical points* of K_g and it is to this technical issue that most of the work in this paper is devoted.

Now we give a quick definition of the symplectic structure $\bar{\Omega}$. It will be useful to think of G^{2r} as a subset of G^{4r} via the map

$$\Phi : G^{2r} \to G^{4r} : (a_1, b_1, \ldots, a_r, b_r) \mapsto (a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_r, b_r, a_r^{-1}, b_r^{-1}).$$

For any $1 \le i \le 4r$, and $x \in G^{4r}$, we write

$$f_i = \mathrm{Ad}(x_{i-1}, \ldots, x_1) : LG \to LG,$$

with f_1 being the identity map. Next let $\tilde{\Omega}$ be the two-form on G^{4r} specified by

$$\tilde{\Omega}_x(xH, xH') = \frac{1}{2} \sum_{1\le i,j\le 4r} \epsilon_{ij} \left\langle f_{i-1}^{-1} H_i, f_{j-1}^{-1} H_j' \right\rangle_{LG},$$

where $H = (H_1, \ldots, H_{4r})$, $H' = (H_1', \ldots, H_{4r}') \in (LG)^{4r}$, and ϵ_{ij} is equal to 1 for $i < j$, is equal to -1 if $i > j$, and is 0 if $i = j$. Finally, define

$$\Omega = \Phi^* \tilde{\Omega}, \quad \text{a two-form on } G^{2r}. \tag{8}$$

The quotient space $\mathcal{M}_g^0 = K_g^{-1}(e)^0/G$, if non-empty, has a unique smooth manifold structure for which the quotient map $q : K_g^{-1}(e)^0 \to K_g^{-1}(e)^0/G$ is a submersion. The restriction of Ω to $K_g^{-1}(e)^0$ drops down to a two-form $\bar{\Omega}$ on $\mathcal{M}_g^0 = K_g^{-1}(e)^0/G$:

$$q^*\bar{\Omega} = \Omega|K_g^{-1}(e)^0. \tag{9}$$

It was shown in [11,12] (with more details in [19]) that $\bar{\Omega}$ is a *symplectic* form on $\mathcal{M}_g^0$, and, as proved in [19] is induced by Marsden–Weinstein-style from the Atiyah–Bott symplectic structure [1] on the space of all connections.

1.3. Other remarks

We take this opportunity to correct in this paper Corollary 3.2 and Lemma 4.4(ii) of [22]. The correct forms involve sets of *full measure* and we have stated the correct result here as Proposition 7. It is this form, using sets of full measure, which is useful both for the results of Sengupta [22] and for our results here. I am very grateful to an anonymous referee for pointing out this error which was present in an earlier version of this paper.

It should be noted that what we compute is the volume of $\mathcal{M}_g^0$ and not of the full moduli space $\mathcal{M}_g$. The latter is not, in general, a smooth manifold but is believed to be the union of symplectic manifolds, called *symplectic strata*, of different dimensions, these manifolds corresponding to the different isotropy groups for the action of G on $K_g^{-1}(e)$. Volumes of all the strata have been calculated for $G = SU(2)$ and $SO(3)$ [21].

2. Summary of technical tools

In this section we collect together some results, proved elsewhere, which we will need.

2.1. A determinant identity

Let V and W be finite-dimensional *real* inner-product spaces, and $A : V \to W$ a linear map. If $A : V \to W$ ($\neq 0$) is a linear isomorphism onto its image $A(V)$, then by the *determinant* of A we shall mean

$$\det A = \text{the determinant of a matrix of } A \text{ relative to orthonormal bases in } V \text{ and } A(V). \tag{10}$$

If $\ker(A) \neq \{0\}$, or if $V = \{0\}$, then we define $\det(A) = 0$.

Thus $\det A$ is determined up to a sign ambiguity, and $|\det A|$ is independent of the choice of bases.

Let $A : V \to W$ and $B : W \to Z$ be linear maps between finite-dimensional inner-product spaces. *If A is an isomorphism onto W or if B is an isometry* (in which case $|\det B| = 1$ unless $W = \{0\}$) then

$$|\det(BA)| = |\det(B)||\det(A)|. \tag{11}$$

Consideration of matrices shows that

$$\det(A|(\ker A)^\perp) = \det(A^*|\mathrm{Im}\, A).$$

The following is a slightly sharpened form of Proposition 2.1 of [22]. (It is this sharper statement which was used in [22].)

Proposition 1. *Let* X, Y $(\neq \{0\})$ *be finite-dimensional real vector spaces equipped with inner-products, and let* V *be a subspace of* X, *and* Z *a subspace of* Y. *Let* $L_1 : X \to Z$ *and* $L_2 : X \to Y$ *be linear maps such that*

$$L_1|V^\perp = 0 \quad \textit{and} \quad L_2|V = 0. \tag{12}$$

Let

$$L = L_1 + L_2, \tag{13}$$

and $N = \ker(L)$. *Then*:

(i) *there exists a*

$$\textit{unitary isomorphism} \quad I : N \oplus N^\perp \to V \oplus V^\perp \quad \textit{and a}$$

$$\textit{linear isomorphism} \quad J : Z \oplus Y \to Z \oplus Y \;\; \textit{with} \;\; |\det J| = 1,$$

such that

$$J((L_1|V) \oplus (L_2|V^\perp))I = (L_1|N) \oplus (L|N^\perp). \tag{14}$$

(ii) *The maps* $L_1|V : V \to Z$ *and* $L_2|V^\perp : V^\perp \to Y$ *are both surjective if and only if* $L_1|N : N \to Z$ *and* $L|N^\perp : N^\perp \to Y$ *are both surjective*.

(iii) *The following equality of determinants holds*:

$$|\det L_1^*||\det L_2^*| = |\det(L_1|N)^*||\det L^*|. \tag{15}$$

Here $L_1^* : Z \to X$, $L_2^* : Y \to X$, $(L_1|N)^* : Z \to N$ *and* $L^* : Y \to X$.

Since the statement is slightly sharper than the one in [22] (where this sharper form is used) we include the full proof, though it is almost identical to that given in [22].

Proof.

(i) Let

$$I : N \oplus N^\perp \to V \oplus V^\perp : (a, b) \mapsto ((a+b)_V, (a+b)_{V^\perp}),$$

wherein the subscripts signify orthogonal projections onto the corresponding subspaces. Since $N \oplus N^\perp \simeq X \simeq V \oplus V^\perp$ isometrically, by means of $(x, y) \mapsto x + y$, I corresponds to the identity map on X and is thus a unitary isomorphism.

Let $L^l : Y \to N^\perp \subset X$, be a linear left-inverse for the injective map $L/N^\perp$; thus $L^l L(b) = b$ for every $b \in N^\perp$. Next define

$$J = J_2 J_1 : Z \oplus Y \to Z \oplus Y,$$

where

$$J_1 : Z \oplus Y \to Z \oplus Y : (a, b) \mapsto J_1(a, b) = (a, a + b),$$
$$J_2 : Z \oplus Y \to Z \oplus Y : (a, b) \mapsto J_2(a, b) = (a - L_1 L^l b, b).$$

It is clear that both J_1 and J_2 are injective. Moreover, they are also surjective, because for any $(z, y) \in Z \oplus Y$, $J_1(z, y - z) = (z, y)$ and $J_2(z + L_1 L^l y, y) = (z, y)$; note that $z + L_1 L^l y \in Z$ because $L_1(X) \subset Z$. So J_1 and J_2 are isomorphisms and hence so is J.

By considering matrix representations for J_1 and J_2, we have $|\det J_1| = |\det J_2| = 1$, and so

$$|\det J| = |\det J_2||\det J_1| = 1. \tag{16}$$

For any $(a, b) \in N \oplus N^\perp$, we have:

$$\begin{aligned} &J((L_1|V) \oplus (L_2|V^\perp))I(a, b) \\ &\quad = J(L_1(a + b)_V, L_2(a + b)_{V^\perp}) = J(L_1(a + b), L_2(a + b)) \\ &\quad = J_2(L_1(a + b), L(a + b)) = J_2(L_1(a + b), L(b)) \\ &\quad = (L_1(a + b) - L_1 L^l L(b), L(b)) = (L_1(a), L(b)). \end{aligned}$$

This proves Eq. (14), and part (i).

(ii) Follows directly from (i).

(iii) Since $L_1|V^\perp = 0$ it follows that $L_1^*(Z) \subset V$. Similarly, $L_2^*(Y) \subset V^\perp$ and $L^*(Y) \subset N^\perp$. So, with appropriately restricted codomains (for instance we are taking $L_1^* : Z \to V$ instead of $L_1^* : Z \to X$):

$$(L_1|V)^* = L_1^*, \qquad (L_2|V^\perp)^* = L_2^*, \quad (L|N^\perp)^* = L^*.$$

In view of this, we may take adjoints in Eq. (14) to obtain:

$$I^*(L_1^* \oplus L_2^*)J^* = (L_1|N)^* \oplus L^* \quad \text{as maps } Z \oplus Y \to N \oplus N^\perp,$$

wherein again some of the operators are taken with restricted codomains. Taking determinants (which, by our definition, is not affected by restriction of codomains), and using the determinant of products given in (11), and the fact that $|\det J|$ is equal to 1, we obtain the determinant formula (15). □

We will use the preceding proposition in a specific context. Let G be a compact, connected, semisimple Lie group with Lie algebra LG equipped with an Ad-invariant metric. Let g_1 and g_2 be positive integers, and $g = g_1 + g_2$. We have the product commutator maps $K_{g_i} : G^{2g_i} \to G$ and $K_g : G^{2g} \to G$ specified through (2). Let $x_i \in G^{2g_i}$ and $x = (x_1, x_2)$. Define

$$C_1 : G^{2g} \to G : (x_1, x_2) \mapsto K_{g_1}(x_1), \qquad C_2 : G^{2g} \to G : (x_1, x_2) \mapsto K_{g_2}(x_2).$$

Then $K_g(x) = C_2(x)C_1(x)$ and we have the derivative maps

$$K_g(x)^{-1}\, dK_g(x) : T_x G^{2g} \to LG, \qquad C_i(x)^{-1}\, dC_i(x) : T_x G^{2g} \to LG,$$

which are related by

$$K_g(x)^{-1}\,\mathrm{d}K_g(x) = C_1(x)^{-1}\,\mathrm{d}C_1(x) + \mathrm{Ad}(C_1(x)^{-1})C_2(x)^{-2}\,\mathrm{d}C_2(x).$$

We will apply Proposition 1 with

$$X - (LG)^{2g} \simeq (LG)^{2g_1} \oplus (LG)^{2g_2}, \qquad V = (LG)^{2g_1} \oplus 0,$$

and

$$L_1 = C_1(x)^{-1}\,\mathrm{d}C_1(x), \qquad L_2 = \mathrm{Ad}(C_1(x)^{-1})C_2(x)^{-1}\,\mathrm{d}C_2(x).$$

Specializing Proposition 1 to this situation gives us the following proposition.

Proposition 2. *Let $x = (x_1, x_2) \in G^{2g_1} \times G^{2g_2}$. Then K_{g_i} is submersive at x_i, for both $i = 1$ and $i = 2$, if and only if K_g is submersive at x and $C_1|K_g^{-1}(e) : K_g^{-1}(e) \to G$ is submersive at x. Furthermore*

$$\begin{aligned}&|\det \mathrm{d}K_g(x)^*||\det[\mathrm{d}C_1(x)|\ker \mathrm{d}K_g(x)]^*|\\ &\quad = |\det \mathrm{d}C_1(x)^*||\det \mathrm{d}C_2(x)^*| = |\det \mathrm{d}K_{g_1}(x_1)^*||\det \mathrm{d}K_{g_2}(x_2)^*|.\end{aligned} \tag{17}$$

2.2. A disintegration formula

The following disintegration formula, proved in Proposition 3.1 of [22] will be useful. (The formula (19) is proved for vastly more general K by Federer [7].)

Proposition 3. *Let $K : M \to N$ be a smooth mapping between Riemannian manifolds. Let $N_K = K(M \setminus C_K)$, where C_K is the set of points where K is not submersive, i.e. the rank of* dK *is less than* dim N. *Assume that $C_K \neq M$. Suppose ϕ is a continuous function of compact support on M. Let* vol *denote Riemannian volume measure.* (*For example, on the submanifold $K^{-1}(h) \setminus C_K \subset M$, for $h \in N_K$, which is given the metric induced from M. If* dim $K^{-1}(h) = 0$, *the Riemannian volume is understood to be counting measure.*)

If ϕ vanishes in a neighborhood of C_K, then

$$h \mapsto \int_{K^{-1}(h)\setminus C_K} \phi\,\mathrm{d}\,\mathrm{vol} \ \ \text{is continuous on } N_K, \tag{18}$$

and

$$\int_M \phi\,\mathrm{d}\,\mathrm{vol} = \int_{N_K}\left[\int_{K^{-1}(h)\setminus C_K} \frac{\phi}{|\det(\mathrm{d}K)^*|}\mathrm{d}\,\mathrm{vol}\right]\mathrm{d}\,\mathrm{vol}(h). \tag{19}$$

In our application, every open subset U of M can be expressed as the union of a sequence of open subsets U_n with compact closure, and there is a sequence of continuous functions $0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_n \uparrow 1_U$, where ϕ_n is 0 outside U_n. Then, for f any continuous non-negative function on M, using $\phi_n f$ in place of f in (19), and letting $n \to \infty$, monotone convergence shows that (19) holds for $f 1_U$ in place of ϕ, if U is any open subset of $M \setminus C_K$. In particular, (19) holds for 1_U in place of ϕ and hence, if $\mathrm{vol}(M) < \infty$, also for 1_{U-V} for any open sets $U, V \subset M \setminus C_K$.

2.3. Some dense sets of full measure

We shall describe some useful subsets which are dense and of full measure in appropriate sets.

The group G acts by conjugation on G^{2r}:

$$G \times G^{24} \to G^{2r} : (h, x) \mapsto hxh^{-1} = (hx_1h^{-1}, \dots, hx_{2r}h^{-1}), \tag{20}$$

where $x = (x_1, \dots, x_{2r})$.

Semisimplicity of the compact group G (i.e. that the center G is finite) is important in the following. We equip G with an Ad-invariant metric.

Proposition 4. *Let G be a compact, connected, semisimple Lie group and T a maximal torus in G, acting on G by conjugation. Then the set of points in G where the isotropy is $Z(G)$ is an open set of full measure.*

By "full measure" we mean a measurable set whose complement has measure zero. In particular, an open set of full measure is automatically dense since the measures under consideration assign positive measure to non-empty open sets.

Proof. Under the adjoint action of the compact abelian group T, the Lie algebra LG splits up as a direct sum of LT and two-dimensional spaces $R_1, \dots, R_k$ on each of which T acts by 'rotations'.

The compact Lie group G, equipped with the Ad-invariant metric on the Lie algebra LG, is a complete Riemannian manifold. We shall use a result concerning the exponential map for such manifolds.

For each unit vector $u \in LG$ let $\delta(u)$ be the infimum of all real numbers $r > 0$ such that the distance of $\exp(ru)$ from the identity e is r. Now let B be the subset of LG consisting of 0 and all $v \neq 0$ such that $|v| < \delta(v/|v|)$, and let $W = \exp(B)$. Then it is known (see, for instance [5, Theorem 3.2 and Proposition 3.1]) that B is open, W is an open set of full measure in G, and

$$B \to W : v \mapsto \exp(v) \text{ is a diffeomorphism onto } W. \tag{21}$$

For any $t \in T$, the conjugation map $G \to G : x \mapsto txt^{-1}$ is an isometric isomorphism and so the function δ is invariant under the adjoint action of T on LG. Therefore, $\mathrm{Ad}(t)B = B$ for all $t \in B$.

Let

$$W^0 = \exp(W'), \tag{22}$$

where W' is the subset of B consisting of all points of the form $v = v_{LT} + v_1 + \cdots + v_k$, with $v_{LT} \in LT$ and each $v_i \in R_i$ being *non-zero*:

$$W' = \{v_{LT} + v_1 + \cdots + v_k | v_{LT} \in LT, \ \text{ each } \ v_i \in R_i \ \text{ and } \ v_i \neq 0\}. \tag{23}$$

Suppose $t \in T$ commutes with $x \in W^0$. We know that $x = \exp(v)$ for a unique $v \in W'$. Moreover, since exp is injective on B and $\mathrm{Ad}(t)v \in B$, the relation $txt^{-1} = x$ implies that

$\mathrm{Ad}(t)v = v$. Since $\mathrm{Ad}(t)$ preserves each subspace R_i, whose *direct sum* along with LT is LG, it follows that $\mathrm{Ad}(t)v_i = v_i$ for each $i \in \{1, \ldots, k\}$. Since T acts on the two-dimensional spaces R_i by rotations and fixes the non-zero vector v_i, this means that in fact $\mathrm{Ad}(t)$ is actually the identity on each R_i. Therefore, $\mathrm{Ad}(t)$ is the identity on all of LG and so $t \in Z(G)$. Thus the T-isotropy at each point of W^0 is $Z(G)$. Now W' is clearly an open subset of full measure in B, and so, since exp is a diffeomorphism on B, it follows that W^0 is a subset of full measure in W. Since W is of full measure in G, we conclude that W^0 is of full measure in G.

By a general result of transformation group theory [2, IX.96, No. 4, Theorem 2; 3, Theorem 4.3.1 and Corollary 6.2.5; 10, Theorem 4.27] for compact Lie groups acting on connected manifolds, the set of points of minimal isotropy is (dense and) open in the whole space. □

We apply this to show that the conjugation action of G on G^r has minimal isotropy $Z(G)$ on a set of full measure.

Proposition 5. *Let G be a compact, connected, semisimple Lie group, and k any integer ≥ 2. For the conjugation action of G on G^k, the subset on which the isotropy group is $Z(G)$ is a dense open set of full measure in G.*

Proof. As noted earlier, the set of points of minimal isotropy (for a compact Lie group acting on a connected manifold) is open, being a consequence of a general result on transformation groups [2, IX.96, No. 4, Theorem 2]. So we focus on the measure theoretic issue.

Since $G^k = G^2 \times G^{k-2}$, it will suffice to prove the result for $k = 2$. Let U be the subset of G^2 consisting of all points where the isotropy group of the conjugation action of G is $Z(G)$. The subset G_0 of G which consists of points which generate maximal tori is of full measure in G (see, for example, [4, Theorem IV.2.11(ii)]). If $x \in G_0$ then the preceding lemma implies that for almost every $y \in G$ the isotropy group at (x, y) is $Z(G)$ (any element which commutes with x lies in the maximal torus generated by x; see, for example [4, Theorem IV.2.3(i)]). So, by Fubini's theorem, $(G_0 \times G) \cap U$ is of full measure in G^2. So U is of full measure in G^2. □

The preceding result has the following consequence.

Proposition 6. *For any integer $r \geq 1$ and compact, connected semisimple group G, the critical points of the mapping $K_r : G^{2r} \to G$ form a set of measure 0 in G^{2r}.*

Proof. There is a remarkable relationship, stated in (32), between the derivative $\mathrm{d}K_r$ and the isotropy of the conjugation action of G on G^{2r}. The relation (32) implies that at any critical point x of K_r the isotropy group $\{g \in G : gxg^{-1} = x\}$ has a non-trivial Lie algebra, and so, in particular, the isotropy group is not equal to $Z(G)$. The preceding proposition then implies that the set of critical points of K_r is contained in a set of measure 0. □

Next we show that almost every point on almost every level set $K_r^{-1}(h)$ is a point of isotropy $Z(G)$. For this we use the important fact that the product commutator map K_r :

$G^{2r} \to G$ is *surjective*. This is proved in [20, Proposition 4.2.4] and uses semisimplicity of G (as I learnt later, this result also appears in [2, Lie IX.33 Corollaire to Proposition 10]).

Proposition 7. *For any integer $r \geq 1$, let $\mathcal{U}_r^0$ be the subset of G^{2r} where the isotropy of the conjugation action of G is $Z(G)$. Then for almost every $h \in G$ the set $K_r^{-1}(h) \cap \mathcal{U}_r^0$ is of full measure in $K_r^{-1}(h)$.*

Proof. Let $\mathcal{U}_r$ be the set of all non-critical points of K_r. Then

$$\mathcal{U}_r^0 \subset \mathcal{U}_r, \tag{24}$$

because of the striking relation (32) between the behavior of $\mathrm{d}K_r$ and the isotropy of the conjugation action. The mapping $K_r|\mathcal{U}_r : \mathcal{U}_r \to G$ is an open mapping. We have the co-area/disintegration formula giving the volume of any open set $A \subset \mathcal{U}_r$:

$$\mathrm{vol}(A) = \int_{K_r(\mathcal{U}_r)} \left[\int_{K_r^{-1}(h)\cap A} \frac{\mathrm{d\,vol}}{|\det(\mathrm{d}K_r)^*|} \right] \mathrm{d\,vol}, \tag{25}$$

where vol always denotes Riemannian volume arising, in our situation, from any choice of Ad-invariant metric on G. Since $\mathrm{vol}(G^{2r}) < \infty$, the formula (25) holds when A is the difference of open sets. Taking A to be the set $\mathcal{U}_r - \mathcal{U}_r^0$ of measure 0, it follows that for almost every $h \in K_r(\mathcal{U}_r)$ the set $K_r^{-1}(h) \cap \mathcal{U}_r^0$ is of full measure in $K_r^{-1}(h) \cap \mathcal{U}_r$. Now $K_r(\mathcal{U}_r)$ contains all regular values of K_r: here we use the *surjectivity* of K_r which assures that every regular value of K_r is in fact a value of K_r. Moreover, by Sard's theorem, the set of all regular values of K_r is a set of full measure in G, and, furthermore, note that $K_r^{-1}(h) \subset \mathcal{U}_r$ for any regular value h of K_r. Thus almost every point $h \in G$ satisfies the condition that $K_r^{-1}(h) \cap \mathcal{U}_r^0$ is of full measure in $K_r^{-1}(h) \cap \mathcal{U}_r = K_r^{-1}(h)$. □

Using the notation from the preceding result we also have the following proposition.

Proposition 8. *For $g_1, g_2 \geq 1$ and $g = g_1 + g_2$, let*

$$K_g^{-1}(e)^{0,0} = (\mathcal{U}_{g_1}^0 \times \mathcal{U}_{g_2}^0) \cap K_g^{-1}(e), \tag{26}$$

and let $C_i : G^{2g_1} \times G^{2g_2} \to G : (x_1, x_2) \mapsto K_{g_i}(x_i)$, for $i \in \{1, 2\}$. Then $K_g^{-1}(e)^{0,0}$ is not empty and the set

$$U_{12} \stackrel{\mathrm{def}}{=} C_1(K_g^{-1}(e)^{0,0}) = C_2(K_g^{-1}(e)^{0,0}) = K_{g_1}(\mathcal{U}_{g_1}^0) \cap K_{g_2}(\mathcal{U}_{g_2}^0), \tag{27}$$

is a dense open subset of full measure in G.

Proof. Let D_i be the set of all regular values of K_{g_i}. If $h \in D_i$ is in the complement of $K_{g_i}(\mathcal{U}_{g_i}^0)$ then $K_{g_i}^{-1}(h) \cap \mathcal{U}_{g_i}^0 = \emptyset$, while, by surjectivity of K_{g_i}, the level set $K_{g_i}^{-1}(h)$ is a non-empty closed submanifold of G^{2g_i} and so has positive volume. So by the preceding result, the set of all such elements h has measure 0. Thus $K_{g_i}(\mathcal{U}_{g_i}^0) \cap D_i$ is of full measure in D_i. By Sard's theorem, D_i is a set of full measure in G, and so $K_{g_i}(\mathcal{U}_{g_i}^0)$ has full measure in G. Since K_{g_i} is submersive on $\mathcal{U}_{g_i}^0$ it follows that the image $K_{g_i}(\mathcal{U}_{g_i}^0)$ is an open subset of G. So the sets $K_{g_i}(\mathcal{U}^0_{g_i})$, for $i \in \{1, 2\}$, are open sets of full measure on G and hence so is their intersection

$$U = K_{g_1}(\mathcal{U}^0_{g_1}) \cap K_{g_2}(\mathcal{U}^0_{g_2}).$$

The relation

$$K_r(b_r, a_r, \dots, b_1, a_1) = K_r(a_1, b_1, \dots, a_r, b_r)^{-1}, \tag{28}$$

shows that

$$K_r(\mathcal{U}^0_r) = K_r(\mathcal{U}^0_r)^{-1},$$

and so

$$U = U^{-1}.$$

Let $h \in U$. Then there is, for $i = 1, 2$, an $x_i \in \mathcal{U}^0_{g_i}$ with $K_{g_1}(x_1) = h$ and $K_{g_2}(x_2) = h^{-1}$. Then $x = (x_1, x_2)$ is a point in $K_g^{-1}(e)^{0,0}$ whose image under C_1 is h and whose image under C_2 is h^{-1}. This, together with the inversion property (28) implies

$$C_i(K_g^{-1}(e)^{0,0}) \supset U,$$

for $i = 1, 2$.

Conversely, suppose $h \in C_1(K_g^{-1}(e)^{0,0})$. This means that there is a point $(x_1, x_2) \in K_g^{-1}(e)^{0,0}$ with $C_1(x_1, x_2) = h$. Since $K_g(x_1, x_2) = C_2(x_2)C_1(x_1)$, it follows that $C_2(x_2) = h^{-1}$. The condition $(x_1, x_2) \in K_g^{-1}(e)^{0,0}$ says also that $x_i \in \mathcal{U}^0_{g_i}$, for $i = 1, 2$, and so $h \in K_{g_1}(\mathcal{U}^0_{g_1})$ and $h^{-1} \in K_{g_2}(\mathcal{U}^0_{g_2})$. The inversion property (28) then implies that $h \in U$. The argument works if we start with $h \in C_2(K_g^{-2}(e)^{0,0})$. □

2.4. Facts about Ω and $\bar{\Omega}$

The compact, semisimple group G acts by conjugation on G^{2g}. Let $\mathcal{U}^0_g$ be the set of all points where the isotropy is $Z(G)$. Clearly, this is carried into itself by the conjugation action. Moreover, $\mathcal{U}^0_g$ is a dense open subset of full measure in G^{2g}, as we have shown.

Let $K_g^{-1}(e)^0 = \mathcal{U}^0_g \cap K_g^{-1}(e)$, the set of points on $K_g^{-1}(e)$ where the isotropy group of the conjugation action of G is $Z(G)$, and assume that it is non-empty (Proposition 8 implies that this is so when $g \geq 2$).

Let $K_g^{-1}(e)_0$ be the set of points x in $K_g^{-1}(e)$ where K_g is submersive i.e. $\mathrm{d}K_g(x) : T_xG^{2g} \to T_{K_g}(x)G$ is surjective. It is a consequence of Theorem 2(v) that $K_g^{-1}(e)^0$ is a subset of $K_g^{-1}(e)_0$.

Then $K_g^{-1}(e)^0$, being a level set of a smooth submersion $K_g|\mathcal{U}^0_g : \mathcal{U}^0_g \to G$, is a smooth submanifold of G^{2g}.

The quotient

$$\mathcal{M}^0_g = \frac{K_g^{-1}(e)^0}{G},$$

being a quotient of a smooth manifold by a compact Lie group, having the same isotropy subgroup $Z(G)$ everywhere, is a smooth manifold (Sections 16.14.1 and 16.10.3 in [6]).

The conjugation action of the group G on G^{2g}, gives for any $x = (x_1, \dots, x_{2g}) \in G^{2g}$ the orbit map

$$\gamma_x : G \to G^{2g} : h \mapsto hxh^{-1} = (hx_1h^{-1}, \dots, hx_{2g}h^{-1}). \tag{29}$$

The derivative at x of the product commutator map $K_g : G^{2g} \to G$ is, technically, a map $T_xG^{2g} \to T_{K_g(x)}G$, but by means of appropriate left translations to the identity we shall sometimes view it as a map $(LG)^{2g} \to LG$ and sometimes as $(LG)^{2g} \to T_{K_g(x)}G$. Its adjoint, relative to the given Ad-invariant metric on LG, is then a linear map

$$\mathrm{d}K_g(x)^* : LG \to (LG)^{2g}. \tag{30}$$

Recall from (8) the two-form Ω on G^{2g}.

We summarize some facts about Ω, γ, and K_g.

Theorem 2. *Let $g \geq 1$ and assume that $K_g^{-1}(e)^0$ is not empty. Then*:

(i) *there is a unique smooth manifold structure on $\mathcal{M}_g^0 = K_g^{-1}(e)^0/G$ such that the quotient map $q : K_g^{-1}(e)^0 \to K_g^{-1}(e)^0/G$ is a submersion*;

(ii) *there is a unique smooth two-form $\bar{\Omega}$ on $K_g^{-1}(e)^0/G$ such that $q^*(\bar{\Omega}) = \Omega | K_g^{-1}(e)^0$*;

(iii) *the two-form $\bar{\Omega}$ is closed and non-degenerate, i.e. it is symplectic on $\mathcal{M}_g^0$* (Proposition IV.E in [11] and [12, Proposition 3.3]);

(iv) *Ω satisfies the "moment map" formula*

$$\Omega_x(xY, \gamma_x' H) = \langle Y, \mathrm{d}K_g(x)^* H\rangle_{(LG)^{2g}}, \tag{31}$$

for all $x \in K_g^{-1}(e)$, $H \in LG$ and $Y \in (LG)^{2g}$ [11, Proposition IV.G];

(v) *for any $x = (x_1, \dots, x_{2g}) \in G^{2g}$, the kernel of $\gamma_x' : LG \to (LG)^{2g}$ is equal to the kernel of $\mathrm{d}K_g(x)^* : LG \to (LG)^{2g}$*:

$$\ker \gamma_x' = \ker \mathrm{d}K_g(x)^* = \{H \in LG : \mathrm{Ad}(x_1)H = \cdots = \mathrm{Ad}(x_{2g})H = H\} \tag{32}$$

([11, Proposition IV.C] and also in [9]);

(vi) *if $x \in K_g^{-1}(e)^0$ then*

$$|\mathrm{Pfaff}(\bar{\Omega}_{q(x)})| = \frac{|\det \gamma_x'|}{|\det dK_g(x)^*|}, \tag{33}$$

where the Pfaffian is, as usual, the square root of the determinant of the matrix of $\bar{\Omega}_{q(x)}$ relative to an orthonormal basis [12, Proposition 3.3];

(vii) *if f is a measurable function on $K_g^{-1}(e)^0$, invariant under the conjugation action of G, and $\tilde{f}$ the induced function on $\mathcal{M}_g^0 = K_g^{-1}(e)^0/G$ then*

$$\int_{\mathcal{M}_g^0} \tilde{f}\, \mathrm{d}\,\mathrm{vol}_{\bar{\Omega}} = \frac{1}{\mathrm{vol}(G/Z(G))} \int_{K_g^{-1}(e)^0} \frac{f}{|\det \mathrm{d}K_g^*|} \mathrm{d}\,\mathrm{vol}, \tag{34}$$

whenever either side is defined, where $\mathrm{vol}_{\bar{\Omega}}$ *is symplectic volume for the symplectic structure* $\bar{\Omega}$, *while* vol *by itself always denotes Riemannian volume*. (*Essentially* [12, Proposition 3.5] *or by part* (vi) *and* [22, Lemma 3.4].)

2.5. An application

We shall "prefabricate" a result that will go into the proof of Theorem 1.

Let g_1, g_2 be positive integers and $g = g_1 + g_2$. Let $K_g^{-1}(e)^{0,0}$ the subset of $K_g^{-1}(e)$ consisting of all points $(x_1, x_2) \in G^{2g_1} \times G^{2g_2}$ such that the isotropy of the G-conjugation action on G^{g_i} is $Z(G)$ at x_i, for $i = 1, 2$. We have the maps $C_i : G^{2g} \to G$ specified by

$$C_1(x_1, x_2) = K_{g_1}(x_1), \qquad C_2(x_1, x_2) = K_{g_2}(x_2).$$

Recall from Proposition 8 (Eq. (27)) that

$$U_{12} \stackrel{\text{def}}{=} C_1(K_g^{-1}(e)^{0,0}) = C_2(K_g^{-1}(e)^{0,0}) = K_{g_1}(\mathcal{U}^0_{g_1}) \cap K_{g_2}(\mathcal{U}^0_{g_2}),$$

is an open subset of full measure in G.

Let D_i be the set of all regular values of K_{g_i}. By Sard's theorem, D_i is a subset of full measure in G. The maps K_{g_i} being surjective, D_i is contained in the image of K_{g_i}. (The set D_i is also open in G.)

The inversion relation

$$K_r(b_r, a_r, \dots, b_1, a_1) = K_r(a_1, b_1, \dots, a_r, b_r)^{-1}, \tag{35}$$

implies that $D_i = D_i^{-1}$. Therefore,

$$D \stackrel{\text{def}}{=} D_1 \cap D_2^{-1}, \tag{36}$$

is also a subset of full measure in G.

Proposition 9. *The following disintegration formula holds*:

$$\int_{K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \mathrm{vol}(G) \int_D \left[\int_{K_{g_1}^{-1}(h)} \frac{\mathrm{d}\,\mathrm{vol}(x_1)}{|\det \mathrm{d}K_{g_1}(x_1)^*|}\right]\left[\int_{K_{g_2}^{-1}(h^{-1})} \frac{\mathrm{d}\,\mathrm{vol}(x_2)}{|\det \mathrm{d}K_{g_2}(x_2)^*|}\right] dh, \tag{37}$$

where dh is the unit-mass Haar measure on G and $\mathrm{vol}(G)$ *is the volume of G with respect to the given Ad-invariant metric on the Lie algebra of G*.

Proof. Let $\mathcal{U}_r^0$ be the subset of G^{2r} consisting of all points where the isotropy of the conjugation action of G is $Z(G)$. Then $\mathcal{U}_r^0$ is a non-empty (in fact, dense) open subset of G^{2r} (this is a special case of a general theorem on group actions: [2, IX.96, No. 4, Theorem 2; 3, Theorem 4.3.1 and Corollary 6.2.5; 10, Theorem 4.27]). By Theorem 2(v), the map $K_g : G^{2g} \to G$ is a submersion at every point in $\mathcal{U}^0_{g_1} \times \mathcal{U}^0_{g_2}$, and so, $K_g^{-1}(e)^{0,0}$, being a level set of a submersion, is a smooth submanifold of G^{2g}.

From Proposition 2 it follows that $C_1|K_g^{-1}(e)^{0,0}$ is submersive at every point. Therefore, by the disintegration formula in Proposition 3, we have

$$\int_{K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \mathrm{vol}(G)\int_{U_{12}}\left[\int_{C_1^{-1}(h)\cap K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*||\det(\mathrm{d}C_1|\ker \mathrm{d}K_g)^*|}\right] dh. \tag{38}$$

Next we use the determinant identity from Proposition 2 to obtain:

$$\int_{K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \mathrm{vol}(G)\int_{U_{12}}\left[\int_{C_1^{-1}(h)\cap K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}(x_1,x_2)}{|\det \mathrm{d}K_{g_1}(x_1)^*||\det \mathrm{d}K_{g_2}(x_2)^*|}\right] dh. \tag{39}$$

Now the identity map

$$C_1^{-1}(h)\cap K_g^{-1}(e)^{0,0} \to K_{g_1}^{-1}(h)^0 \times K_{g_2}^{-1}(h^{-1})^0 : (x_1,x_2)\to(x_1,x_2),$$

is an isometry (the metric on the left is inherited from that on G^{2g}). So we have

$$\int_{K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \mathrm{vol}(G)\int_{U_{12}}\left[\int_{K_{g_1}^{-1}(h)^0} \frac{\mathrm{d}\,\mathrm{vol}(x_1)}{|\det \mathrm{d}K_{g_1}(x_1)^*|}\right]\left[\int_{K_{g_2}^{-1}(h^{-1})^0} \frac{\mathrm{d}\,\mathrm{vol}(x_2)}{|\det \mathrm{d}K_{g_2}(x_2)^*|}\right] dh. \tag{40}$$

Since both U_{12} and D are subsets of full measure in G, the integration $\int_{U_{12}} \cdots dh$ above can be replaced by $\int_D \cdots dh$. Finally, by Proposition 7, the set $K_{g_i}^{-1}(c)^0$ is of full measure in $K_{g_i}^{-1}(c)$ for almost every c, and so we obtain the desired formula (37). □

2.6. A heat-kernel integral and its limit

If $X_1, \ldots, X_d$ is an orthonormal basis of the Lie algebra of G, and α an irreducible representation of G then $\sum_{i=1}^d \alpha_*(X_i)^2$ is of the form $-C_\alpha I$, where C_α is a scalar (Casimir) and I is the identity operator on the representation space of α. The heat-kernel Q_t has a standard character expansion:

$$Q_t(x) = \sum_\alpha (\dim\alpha)\,\mathrm{e}^{-C_\alpha t/2}\chi_\alpha(x),$$

where χ_α is the character of the representation α.

The following is a very useful formula:

$$\int_{G^{2g}} Q_t(hb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1)\,\mathrm{d}a_1\cdots \mathrm{d}b_g = \sum_\alpha \frac{\mathrm{e}^{-C_\alpha t/2}\chi_\alpha(h)}{(\dim\alpha)^{2g-1}}, \tag{41}$$

where the sum is over all inequivalent irreducible representations α of G. This can be verified using: (i) the identity (see Example 4.17.3 in [4])

$$\int_G \chi_\alpha(aba^{-1}c)\,\mathrm{d}a = (\dim\alpha)^{-1}\chi_\alpha(b)\chi_\alpha(c),$$

(ii) repeated application of standard convolution properties of characters, and (iii) commuting integral and a series sum. Integral and sum can be commuted because

$$\sum_\alpha \mathrm{e}^{-C_\alpha t/2}(\dim\alpha)\int|\chi_\alpha(\cdots)|\,\mathrm{d}\cdots \le \sum_\alpha \mathrm{e}^{-C_\alpha t/2}(\dim\alpha)^2 = Q_t(e) < \infty.$$

Formula (41) is by Witten [24, Eq. (2.51)] who determined it in his exact evaluation of the partition function of two-dimensional quantum Yang–Mills theory (the heat-kernel was not used explicitly in [24]).

It is known [13, Lemma 10.3] that $\sum_\alpha(1/(\dim\alpha)^k) < \infty$ for $k \ge 2$. So, for $g \ge 2$, using dominated convergence in (41) gives

$$\lim_{t\downarrow 0}\int_{G^{2g}} Q_t(hb_g^{-1}a_g^{-1}b_ga_g\cdots b_1^{-1}a_1^{-1}b_1a_1)\,\mathrm{d}a_1\cdots\mathrm{d}b_g = \sum_\alpha \frac{\chi_\alpha(h)}{(\dim\alpha)^{2g-1}}. \tag{42}$$

Proposition 10. *The limit formula (42) continues to hold, with the limit* $\lim_{t\downarrow 0}$ *and the sum* $\sum_\alpha$ *being both in the* $L^2(G, dh)$*-sense.*

Proof. Let $k = 2g - 1$, and $d_\alpha = \dim\alpha$. Then

$$\left\|\sum_\alpha \frac{\mathrm{e}^{-C_\alpha t}}{d_\alpha^k}\chi_\alpha - \sum_\alpha \frac{1}{d_\alpha^k}\chi_\alpha\right\|^2_{L^2(G)} = \sum_\alpha \frac{(\mathrm{e}^{-tC_\alpha}-1)^2}{d_\alpha^{2k}},$$

which, for $t > 0$, is bounded, term by term, by the convergent series $\sum_\alpha(1/d_\alpha^{2k})$. □

3. Evaluation of limits

With notation and assumptions as before, let

$$K_g^{-1}(h)_0 \stackrel{\text{def}}{=} \text{the set of all non-critical points of } K_g : G^{2g} \to G \text{ which lie on } K_g^{-1}(h) \tag{43}$$

for any $h \in G$.

A point $x \in G^{2g}$ is a non-critical point of K_g if and only if the isotropy group at x of the conjugation action of G on G^{2g} is discrete, an observation immediate from Theorem 2(v). Therefore, in particular

$$K_g^{-1}(e)^0 \subset K_g^{-1}(e)_0. \tag{44}$$

If $g \ge 2$ then, by Proposition 8 (also Proposition IIIB of [11]), $K_g^{-1}(e)^0$ is not empty and hence also $K_g^{-1}(e)_0 \ne \emptyset$.

As a consequence of the disintegration formula, we have the following result (mentioned in [11, Section IV]).

Lemma 1. *Suppose* g *is an integer* ≥ 2. *Let* f *be a continuous function on* G^{2g} *which is* 0 *in a neighborhood of the critical points of* K_g. *Then*

$$\lim_{t\downarrow 0}\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \mathrm{vol}(G)^{1-2g}\int_{K_g^{-1}(e)_0}\frac{f}{|\det \mathrm{d}K_g^*|}\mathrm{d}\,\mathrm{vol}. \tag{45}$$

Proof. Let C be the set of all critical points of K_g. Then the complement $G^{2g}\setminus C$ is open and the image $K_g(G^{2g}\setminus C)$ is an open subset of G of full measure (by Sard's theorem, since it contains all regular values of the *surjective* map K_g) and hence is also dense in G. By Proposition 3 we have the disintegration

$$\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \mathrm{vol}(G)^{-2g}\int_{K_g(G^{2g}\setminus C)} F(h)Q_t(h)\,\mathrm{d}\,\mathrm{vol}(h), \tag{46}$$

where

$$F(h)\stackrel{\mathrm{def}}{=}\int_{K_g^{-1}(h)_0}\frac{f}{|\det \mathrm{d}K_g^*|}\mathrm{d}\,\mathrm{vol}, \tag{47}$$

is a continuous function of $h\in K_g(G^{2g}\setminus C)$.

The identity e belongs to $K_g(G^{2g}\setminus C)$ since $K_g^{-1}(e)_0\neq\emptyset$. Moreover, $F(h)$ is 0 when h is outside the compact set $K_g(\mathrm{support}(f))\subset K_g(G^{2g}\setminus C)$. So F extends to a continuous function on G, 0 outside $K_g(G^{2g}\setminus C)$. So, remembering that the Riemannian volume on G is $\mathrm{vol}(G)$ times the normalized Haar mass dh

$$\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \mathrm{vol}(G)^{1-2g}\int_G F(h)Q_t(h)dh, \tag{48}$$

and, by the initial condition property of the heat-kernel Q_t, this approaches the limit

$$\mathrm{vol}(G)^{1-2g}F(e) = \mathrm{vol}(G)^{1-2g}\int_{K_g^{-1}(e)_0}\frac{f}{|\det \mathrm{d}K_g^*|}\mathrm{d}\,\mathrm{vol},$$

as $t\downarrow 0$. □

Things are much easier when we deal with a regular value of K_g.

Lemma 2. *Let* r *be any integer* ≥ 1, f *a continuous function on* G^{2r}, *and* c *a regular value of* $K_r : G^{2r}\to G$. *Then*

$$\lim_{t\downarrow 0}\int_{G^{2r}} f(x)Q_t(K_r(x)c^{-1})\,\mathrm{d}x = \mathrm{vol}(G)^{1-2r}\int_{K_r^{-1}(c)}\frac{f}{|\det \mathrm{d}K_r^*|}\mathrm{d}\,\mathrm{vol}. \tag{49}$$

Proof. The argument is essentially the same as in the preceding lemma, but we no longer have to worry about critical points of K_r since there are none on $K_r^{-1}(c)$.

Let U and V be neighborhoods of c, with $\bar{V} \subset U$, and $\bar{U}$ consisting only of regular values of K_r. Let ϕ be a continuous function on G, with $0 \le \phi \le 1$ everywhere, equal to 1 on V and equal to 0 outside U. Let $\psi = 1 - \phi$. Then $f = (\phi \circ K_r) f + (\psi \circ K_r) f$, and

$$\left| \int_{G^{2r}} f(x)\psi(K_r(x)) Q_t(K_r(x)c^{-1})\,\mathrm{d}x \right| \le |f|_{\sup} \sup_{y\in G\setminus V} Q_t(yc^{-1}) \to 0, \quad \text{as } t \downarrow 0,$$

by a uniform-limit property of the heat-kernel Q_t as $t \downarrow 0$.

On the other hand, the integrand in

$$\int_{G^{2r}} f(x)\phi(K_r(x)) Q_t(K_r(x)c^{-1})\,\mathrm{d}x,$$

is 0 near the critical points of K_r. Note also that $\phi(K_r(x)) = 1$ when $x \in K_r^{-1}(c)$, and $K_r^{-1}(c)$ contains no critical point of K_r. So, by Proposition 3 and the argument used in Lemma 1, as $t \downarrow 0$, this integral approaches the limit

$$\mathrm{vol}(G)^{1-2r} \int_{K_r^{-1}(c)} \frac{f}{|\det \mathrm{d}K_r^*|} \mathrm{d}\,\mathrm{vol}.$$

Combining all these observations, we obtain the desired result. □

The preceding result is essentially present in Forman [8].

4. Proof of the main result

Let g be a positive integer. Recall that $K_g^{-1}(e) \subset G^{2g}$. The set of points on $K_g^{-1}(e)$ where $\mathrm{d}K_g(x) : T_x G^{2g} \to T_{K_g(x)} G$ is surjective is denoted as $K_g^{-1}(e)_0$. The set of points on $K_g^{-1}(e)$ where the isotropy group of the G-conjugation action is $Z(G)$ is denoted $K_g^{-1}(e)^0$.

Now suppose g_1 and g_2 are positive integers with $g = g_1 + g_2$. We denote by $K_g^{-1}(e)^{0,0}$ the subset of $K_g^{-1}(e)$ consisting of all points $(x_1, x_2) \in G^{2g_1} \times G^{2g_2}$ such that the isotropy of the G-conjugation action on G^{g_i} is $Z(G)$ at x_i, for $i = 1, 2$. Thus

$$K_g^{-1}(e)^{0,0} = \cup_{c\in G} K_{g_1}^{-1}(c^{-1})^0 \times K_{g_2}^{-1}(c)^0. \tag{50}$$

The subset $\mathcal{U}_{g_i}^0$ of G^{2g_i} where the isotropy group is $Z(G)$ is (dense and) open in G^{2g_i}, as proved in Proposition 5. So

$$K_g^{-1}(e)^{0,0} = (\mathcal{U}_{g_1}^0 \times \mathcal{U}_{g_1}^0) \cap K_g^{-1}(e) = (\mathcal{U}_{g_1}^0 \times \mathcal{U}_{g_2}^0) \cap K_g^{-1}(e)^0,$$

is an open subset of $K_g^{-1}(e)^0$.

Theorem 3. *For any integer $g \ge 2$, and integers $g_1, g_2 \ge 1$ with $g = g_1 + g_2$:*

$$\int_{K_g^{-1}(e)_0} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \int_{K_g^{-1}(e)^0} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|}, \tag{51}$$

$$\int_{K_g^{-1}(e)_0} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \int_{K_g^{-1}(e)^{0,0}} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|}, \tag{52}$$

$$\int_{K_g^{-1}(e)_0} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \mathrm{vol}(G)^{2g-2} \lim_{t\downarrow 0} \int_{G^{2g}} Q_t(K_g(x))\,\mathrm{d}x. \tag{53}$$

Proof. If f is a continuous function on G^{2g}, with $0 \leq f \leq 1$, which is 0 in a neighborhood of the critical points of K_g then

$$\mathrm{vol}(G)^{1-2g} \int_{K_g^{-1}(e)_0} \frac{f\,\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} = \lim_{t\downarrow 0} \int_{G^{2g}} f(x) Q_t(K_g(x))\,\mathrm{d}x \leq \lim_{t\downarrow 0} \int_{G^{2g}} Q_t(K_g(x))\,\mathrm{d}x. \tag{54}$$

The right side was noted in (42) to be finite. Taking appropriate f, with $f = 1$ at distances beyond $1/n$ from the critical points of K_g, and then letting $n \to \infty$ we have, by dominated convergence

$$\mathrm{vol}(G)^{1-2g} \int_{K_g^{-1}(e)_0} \frac{\mathrm{d}\,\mathrm{vol}}{|\det \mathrm{d}K_g^*|} \leq \lim_{t\downarrow 0} \int_{G^{2g}} Q_t(K_g(x))\,\mathrm{d}x. \tag{55}$$

Next, observing that

$$K_g(x_1, x_2) = K_{g_2}(x_2) K_{g_1}(x_1),$$

for $x_1 \in G^{g_1}$ and $x_2 \in G^{g_2}$, and using the convolution property of the heat-kernel

$$\int_G Q_t(ac) Q_s(c^{-1}b)\,\mathrm{d}c = Q_{t+s}(ab) = Q_{t+s}(ba),$$

we have

$$\int_G \left[\int_{G^{2g_1}} Q_t(K_{g_1}(x_1)c^{-1})\,\mathrm{d}x_1 \int_{G^{2g_2}} Q_t(cK_{g_2}(x_2))\,\mathrm{d}x_2\right] \mathrm{d}c = \int_{G^{2g}} Q_{2t}(K_g(x))\,\mathrm{d}x. \tag{56}$$

Then

$$\begin{aligned} &\lim_{t\to 0} \int_{G^{2g}} Q_t(K_g(x))\mathrm{d}x \\ &= \lim_{t\to 0} \int_G \left[\int_{G^{2g_1}} Q_t(K_{g_1}(x_1)c^{-1})\,\mathrm{d}x_1 \int_{G^{g_2}} Q_t(cK_{g_2}(x_2))\,\mathrm{d}x_2\right] \mathrm{d}c \\ &= \int_G \left(\lim_{t\to 0} \int_{G^{2g_1}} \cdots\right) \left(\lim_{t\to 0} \int_{G^{2g_2}} \cdots\right) \mathrm{d}c, \end{aligned} \tag{57}$$

because of the $L^2(G, \mathrm{d}c)$-convergence of the limits $\lim_{t\to 0}$ noted in Proposition 10.

Let D_i be the set of all regular values of $K_{g_i} : G^{2g_i} \to G$, and

$$D \stackrel{\mathrm{def}}{=} D_1 \cap D_2, \tag{58}$$

which, as we have already noted in the context of (36), is a dense open subset of full measure in G.

Since D is of full measure in G, we can replace $\int_G \cdots \mathrm{d}c$ by $\int_D \cdots \mathrm{d}c$ on the right side in (57). Then using the limit value computed in Lemma 2 we have

$$\lim_{t\to 0}\int_{G^{2g}} Q_t(K_g(x))\mathrm{d}x = \mathrm{vol}(G)^{2-2g}\int_D\left[\int_{K_{g_1}^{-1}(c)}\frac{\mathrm{d\,vol}}{|\det \mathrm{d}K_{g_1}^*|}\int_{K_{g_2}^{-1}(c^{-1})}\frac{\mathrm{d\,vol}}{|\det \mathrm{d}K_{g_2}^*|}\right]\mathrm{d}c. \tag{59}$$

Now inserting our "prefabricated" piece Proposition 9, we see that the integral $\int_D[\cdots]\mathrm{d}c$ on the right side in (59) is equal to

$$[\mathrm{vol}(G)]^{-1}\int_{K_g^{-1}(e)^{0,0}}\frac{\mathrm{d\,vol}}{|\det \mathrm{d}K_g^*|}.$$

Combining this with (55), we write

$$\mathrm{vol}(G)^{1-2g}\int_{K_g^{-1}(e)_0}\frac{\mathrm{d\,vol}}{|\det \mathrm{d}K_g^*|} \le \lim_{t\to 0}\int_{G^{2g}} Q_t(K_g(x))\,\mathrm{d}x = \int_{K_g^{-1}(e)^{0,0}}\frac{\mathrm{d\,vol}}{|\det \mathrm{d}K_g^*|}\mathrm{vol}(G)^{1-2g}. \tag{60}$$

Since $K_g^{-1}(e)^{0,0} \subset K_g^{-1}(e)_0$, it follows that the inequalities in (60) are equalities. □

Since the middle integral in (60) is finite so are the others. As a consequence, we have the following corollary.

Corollary 1. *For any integer $g \geq 2$, the sets $K_g^{-1}(e)^{0,0}$ and $K_g^{-1}(e)^0$ open, dense subsets of full measure in $K_g^{-1}(e)_0$.*

Now we are ready for the following proposition.

Proposition 11. *For any integer $g \geq 2$ and any continuous function f on G^{2g}*

$$\lim_{t\downarrow 0}\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \mathrm{vol}(G)^{1-2g}\int_{K_g^{-1}(e)^0}\frac{f}{|\mathrm{d}K_g^*|}\mathrm{d\,vol}. \tag{61}$$

Proof. We have proved this (in Lemma 1) when f is zero near the critical points of K_g. We have also proved this for $f = 1$ in Theorem 3. Now by Proposition 6, the set $\mathcal{U}_g$ of non-critical points of K_g is of full measure in G^{2g}, and so

$$\int_{G^{2g}} f(x)Q_t(K_g(x))\,\mathrm{d}x = \int_{\mathcal{U}_g} f(x)Q_t(K_g(x))\,\mathrm{d}x.$$

Since $K_g^{-1}(e)^0$ is a subset of $\mathcal{U}_g$, the task reduces to proving a limiting result for integrals over $\mathcal{U}_g$, given that the limiting formula holds for continuous functions of compact support as well as for the constant function 1. The proof is finished by using Lemma 3 below (take X to be $\mathcal{U}_g$, which is an open subset of G^{2g}). □

Lemma 3. *Let μ_t, for $t \geq 0$, be finite Borel measures on a locally compact Hausdorff space X such that* $\lim_{t\downarrow 0} \mu_t(X) = \mu_0(X)$ *and*

$$\lim_{t\downarrow 0} \int_X f \,\mathrm{d}\mu_t = \int_X f \,\mathrm{d}\mu_0,$$

for every continuous function f of compact support in X. Assume that X is the union of a countable collection of compact sets. Then

$$\lim_{t\downarrow 0} \int_X f \,\mathrm{d}\mu_t = \int_X f \,\mathrm{d}\mu_0,$$

for every bounded continuous function f on X.

Proof. Let $\epsilon > 0$.

Since X is the union of a countable number of compact sets, and $\mu_0(X) < \infty$, there is a compact set $K \subset X$ for which

$$\mu_0(K^c) < \epsilon.$$

By local compactness there is an open set $U \supset K$ with compact closure $\bar{U}$, and, by Urysohn's lemma, there is a continuous function Φ with

$$1_K \leq \Phi \leq 1_U.$$

First we demonstrate that $\limsup_{t\downarrow 0} \mu_t(\bar{U})$ is $< \epsilon$. For $s > 0$ we have

$$\mu_s(\bar{U}^c) = \mu_s(X) - \mu_s(\bar{U}) \leq \mu_s(X) - \int_X \Phi \,\mathrm{d}\mu_s,$$

and so, for any $t > 0$,

$$\sup_{0<s\leq t} \mu_s(\bar{U}^c) \leq \sup_{0<s\leq t} \mu_s(X) - \inf_{0<s\leq t} \int_X \Phi \,\mathrm{d}\mu_s,$$

which implies

$$\limsup_{t\downarrow 0} \mu_t(\bar{U}^c) \leq \limsup_{t\downarrow 0} \mu_t(X) - \liminf_{t\downarrow 0} \int_X \Phi \,\mathrm{d}\mu_t = \mu_0(X) - \int_X \Phi \,\mathrm{d}\mu_0 < \mu_0(K^c) < \epsilon.$$

Now choose an open set $V \supset \bar{U}$ with compact closure $\bar{V}$, and a continuous function ψ with

$$1_{\bar{U}} \leq 1 - \psi \leq 1_V, \quad \text{i.e.}\ \ 1_{V^c} \leq \psi \leq 1_{\bar{U}^c}. \tag{62}$$

Let f be a continuous function on X and write it as

$$f = \psi f + (1 - \psi) f.$$

Since $(1 - \psi) f$ is continuous and of compact support

$$\lim_{t\downarrow 0} \int_X (1 - \psi) f \,\mathrm{d}\mu_t = \int_X (1 - \psi) f \,\mathrm{d}\mu_0.$$

Now we must bound $\int_X \psi f \, \mathrm{d}\mu_t - \int_X \psi f \, \mathrm{d}\mu_0$. To this end, we have

$$\left| \int_X f\psi \, \mathrm{d}\mu_t \right| \le |f|_{\sup} \mu_t(\bar{U}^c),$$

for all $t \ge 0$.

Combining all this, we have

$$\limsup_{t\downarrow 0} \left| \int_X f \, \mathrm{d}\mu_t - \int_X f \, \mathrm{d}\mu_0 \right| \le \limsup_{t\downarrow 0} [|f|_{\sup} \mu_t(\bar{U}^c) + |f|_{\sup} \mu_0(\bar{U}^c)] \le 2|f|_{\sup} \epsilon,$$

and since $\epsilon > 0$ is arbitrary, this is all we needed. □

Finally, we can turn to the following proof.

Proof of Theorem 1. Let f be a continuous function on G^{2g}, invariant under the conjugation action of G, and $\tilde{f}$ the function induced on $\mathcal{M}_g^0 = K_g^{-1}(e)/G$. Then

$$\lim_{t\downarrow} \int_{G^{2g}} f(x) Q_t(K_g(x)) \, \mathrm{d}x = \mathrm{vol}(G)^{1-2g} \int_{K_g^{-1}(e)^0} \frac{f}{|\mathrm{d}K_g^*|} \mathrm{d}\,\mathrm{vol} \quad \text{(by Eq. (61))}$$

$$= \mathrm{vol}(G)^{1-2g} \frac{\mathrm{vol}(G)}{|Z(G)|} \int_{\mathcal{M}_g^0} \tilde{f} \, \mathrm{d}\,\mathrm{vol}_{\bar{\Omega}} \quad \text{(by Thoerem 2(vii))}$$

which is what we had set out to prove. □

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Appendix A. Background/heuristics

We shall summarize the background from which Theorem 1 arises.

A.1. Geometric terminology

Let Σ be a closed (= compact without boundary), oriented two-dimensional Riemannian manifold, and G a compact, connected, semisimple Lie group with Lie algebra LG equipped with an Ad-invariant metric. Let $\pi : P \to \Sigma$ be a *principal G-bundle*, i.e. P is a smooth manifold with a smooth right action of G on P denoted by

$$P \times G \to P : (p, g) \mapsto pg = R_g p = \gamma_p(g),$$

and $\pi : P \to M$ is a smooth surjection such that each point $m \in M$ has an open neighborhood U for which there is a C^∞ diffeomorphism $\phi : U \times G \to \pi^{-1}(U)$ satisfying $\pi\phi(a, g) = a$ and $\phi(a, g)h = \phi(a, gh)$ for every $(a, g, h) \in U \times G^2$.

A *connection* on P is an LG-valued one-form ω on P for which $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for every $g \in G$, and $\omega(\gamma_p'(H)) = H$ for every $p \in P$ and $H \in LG$. The set $\mathcal{A}$ of all connections on P is an infinite-dimensional affine space. The tangent space $T_\omega\mathcal{A}$ is $\{\omega' - \omega : \omega' \in \mathcal{A}\}$ and this is readily checked to be

$$T_\omega\mathcal{A} = \bar{\Lambda}^1(P; LG),$$

the latter being the set of all smooth one-form α on P with values in LG and satisfying $R_g^*\alpha = \mathrm{Ad}(g^{-1})\alpha$ and $\alpha_p(v) = 0$ for all $g \in G$, $p \in P$, and all $v \in \ker \pi'(p)$.

A *gauge transformation* or *bundle automorphism* is a C^∞ diffeomorphism $\phi : P \to P$ for which $\phi \circ R_g = R_g \circ \phi$ for all $g \in G$ and $\pi \circ \phi = \pi$. The set of all gauge transformations forms a group $\mathcal{G}$ under composition and this group acts on the right on $\mathcal{A}$ by

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A} : (\omega, \phi) \mapsto \phi^*\omega.$$

Physically, elements of $\mathcal{A}$ are *gauge fields* and elements of the quotient space $\mathcal{A}/\mathcal{G}$ represent physically equivalent classes of gauge field configurations. It is mathematically convenient to fix a basepoint $o \in \Sigma$ and work with the subgroup $\mathcal{G}_0$ of $\mathcal{G}$ consisting of all $\phi \in \mathcal{G}$ for which $\phi(u) = u$ for any $u \in \pi^{-1}(o)$, and the corresponding quotient

$$\mathcal{C}_0 = \frac{\mathcal{A}}{\mathcal{G}_o}.$$

For any connection $\omega \in \mathcal{A}$, the *ω-horizontal lift* of a C^1 path $c : [0, 1] \to M$ through any point $u \in \pi^{-1}(c(0))$ is the unique C^1 path $\tilde{c}^\omega : [0, 1] \to P$ for which $\pi \circ \tilde{c}^\omega = c$, $\tilde{c}^\omega(0) = u$, and $\omega((\tilde{c}^\omega)'(t)) = 0$ for all $t \in [0, 1]$. Piecing such paths together extends the notion to piecewise C^1 paths c. If c is a loop then $\tilde{c}^\omega(1)$ is on the same fiber as u and so there is a unique $h \in G$ for which $\tilde{c}^\omega(1) = uh$; this h is the *holonomy* of ω around the loop c, with initial point u:

$$h_u(c; \omega) : \text{holonomy of } \omega \text{ around } c, \text{ with initial point } u.$$

If u is replaced by ug for some $g \in G$ then $h_u(c; \omega)$ gets conjugated by g, while if ω is replaced by $\phi^*\omega$ then $h_u(c; \omega)$ gets conjugated by $\phi(u)$, where $\phi(u)$ is the unique element of G for which $\phi(u) = u\hat{\phi}(u)$. Consequently, if f is any function on G^n which is invariant under the conjugation action of G on G^n, and $c_1, \ldots, c_n$ are piecewise smooth closed loops on Σ based at some point then

$$f(h_u(c_1; \omega) \cdots h_u(c_n; \omega)),$$

is independent of the choice of u and specifies a function on the quotient space $\mathcal{A}/\mathcal{G}$.

The *curvature* Ω^ω of a connection ω is the LG-valued two-form on P given on any vectors $X, Y \in T_pP$ by

$$\Omega^\omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)].$$

A.2. The Euclidean quantum Yang–Mills functional integral

The invariance properties of Ω^ω and the Ad-invariance of the metric on LG implies that there is a well-defined function $|\Omega^\omega|$ on Σ whose value at any point m is equal to $|\Omega^\omega(e_1, e_2)|_{LG}$, where e_1, e_2 are vectors in T_pP projecting by $\pi'(p)$ to an orthonormal basis in T_mM, p being any point in the fiber $\pi^{-1}(m)$. The *Yang–Mills action functional* S_{YM} is the function on $\mathcal{A}$ given by

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2}\int_\Sigma |\Omega^\omega|^2 \,\mathrm{d}\sigma, \tag{A.1}$$

where σ is the area-measure induced by the metric on Σ.

The Euclidean quantum Yang–Mills theory of the gauge fields ω on Σ leads to consideration of integrals

$$\int_{\mathcal{A}} f(h_u(c_1; \omega) \cdots h_u(c_n; \omega))\, \mathrm{e}^{-S_{\mathrm{YM}}(\omega)/t} D\omega, \tag{A.2}$$

where t is a positive parameter, the integrand is the function described before and $D\omega$ is "Lebesgue measure" on $\mathcal{A}$ corresponding to the metric on $\mathcal{A}$ determined by the metrics on Σ and LG. Expression (A.2) is formal since no useful rigorous version of such a "Lebesgue measure" exists for the infinite-dimensional space $\mathcal{A}$.

A.3. The rigorous YM functional integral

In [17] the following rigorous framework was constructed using (A.2) as a guide. View Σ as a quotient:

$$q : D \to \Sigma,$$

where D is the closed unit disk and q pastes together certain pairs of arcs on ∂D. Choose the basepoint $o = q(O)$, where O is the origin in D. Take any triangulation of D made up of radial segments and cross-radial segments, such that D projects to a triangulation T of Σ. In [17] a probability measure μ_t was constructed on a space $\overline{\mathcal{C}_o}$ and for each loop c made up of edges of T a random variable $h(c; \omega)$ was constructed on $\overline{\mathcal{C}_o}$, guided by the goal of realizing the normalized form of the integral (A.2) as

$$\int_{\overline{\mathcal{C}_0}} f(h(c_1; \omega) \cdots h(c_n; \omega))\, \mathrm{d}\mu_t(\omega). \tag{A.3}$$

The value of this rigorously defined integral was calculated.

A.4. The discrete Yang–Mills measure

Let T be any two-dimensional simplicial complex triangulating our surface Σ. Let $E_T = \{e_1, \bar{e}_1, \ldots, e_N, \bar{e}_N\}$ be the set of all oriented one-simplices of T, with $\bar{e}$ denoting the reverse of e. Let $\mathcal{A}_T$ be the set of all $x \in G^{E_T}$, mappings $E_T \to G : b \mapsto x_b = x(b)$, for which $x_{\bar{e}_j} = x_{e_j}^{-1}$ for all edges e_j. If κ is any path made up of edges $\kappa = b_m, \ldots, b_1$ and $x \in \mathcal{A}_T$ define

$$x(\kappa) \stackrel{\text{def}}{=} x(b_m), \ldots, x(b_1).$$

On $\mathcal{A}_T$ there is the unit-mass normalized Haar measure

$$\mathrm{d}x = \mathrm{d}x_{e_1}, \ldots, \mathrm{d}x_{e_N}, \tag{A.4}$$

where each $\mathrm{d}x_{e_j}$ is Haar measure of total mass 1 on G. Now let

$$(0, \infty) \times G \to \mathbf{R} : (s, x) \mapsto Q_s(x),$$

be the heat-kernel on G specified by the metric on G normalized to $\int_G Q_t(y)\,\mathrm{d}y = 1$ where $\mathrm{d}y$ is Haar measure of total mass 1 on G. The *discrete Yang–Mills measure* ν_t^T on $\mathcal{A}_T$ is given by

$$\mathrm{d}\nu_t^T(x) = \prod_{\Delta} Q_{t|\Delta|}(x(\partial\Delta))\,\mathrm{d}x, \tag{A.5}$$

where the product is over all the two-simplices Δ of T, $|\Delta|$ denotes the area enclosed by Δ, and the conjugation/inversion–invariance property of the heat-kernel ensures that $Q_{t|\Delta|}(x(\partial\Delta))$ does not depend on where boundary loop $\partial\Delta$ is based and which way it is oriented. The convolution property (A.16) can be used to show that ν_t^T has an invariance property under subdivisions of the triangulation T (see [17, Chapter 7]). Though we have used a simplicial complex T, we could have worked with a cell-complex.

A.5. The Yang–Mills loop expectations

Assume now that G is simply connected (the general case requires additional issues and notation).

In [17, Theorem 8.4] (see also the introduction in [17] for a statement) it is proved that

$$\int_{\overline{\mathcal{C}_0}} f(h(c_1;\omega)\cdots h(c_n;\omega))\,\mathrm{d}\mu_t(\omega) = \frac{1}{N_t}\int f(x(c_1)\cdots x(c_n))\,\mathrm{d}\nu_t^T(x), \tag{A.6}$$

where N_t is the normalizing factor

$$N_t = \nu_t(\mathcal{A}_T),$$

given explicitly by

$$N_t = \int_{G^{2g}} Q_{t|\Sigma|}(b_g^{-1}a_g^{-1}b_g a_g \cdots b_g^{-1}a_g^{-1}b_g a_g)\,\mathrm{d}a_1\,\mathrm{d}b_1\cdots\mathrm{d}a_g\,\mathrm{d}b_g. \tag{A.7}$$

Here we are assuming that Σ is a closed, oriented surface of genus $g \geq 1$. Note that N_t does not depend on the triangulation T. Heuristically, N_t corresponds to the "partition function" $\int_{\mathcal{A}} \mathrm{e}^{-S_{\mathrm{YM}}(\omega)/t} D\omega$:

$$N_t \sim \int_{\mathcal{A}} \mathrm{e}^{-S_{\mathrm{YM}}(\omega)/t} D\omega. \tag{A.8}$$

A.6. Symplectics

On the infinite-dimensional affine space $\mathcal{A}$ there is a *symplectic structure* Ω, due to Atiyah and Bott, given on any two vectors $A, B \in T_\omega \mathcal{A}$ by

$$\Omega(A, B) = \int_\Sigma \langle A \wedge B \rangle, \tag{A.9}$$

where $\langle A \wedge B \rangle$ is the two-form on Σ whose value on any vectors $X, Y \in T_m \Sigma$ is

$$\langle A \wedge B \rangle (X, Y) = \langle A(X), B(Y) \rangle_{LG} - \langle A(Y), B(Y) \rangle_{LG}.$$

A straightforward calculation (see, for example (5.5b) in [19]) shows that the action of $\mathcal{G}$ on $\mathcal{A}$ preserves this structure and there is a corresponding moment map, this being in fact the curvature function

$$\omega \mapsto J(\omega) = \Omega^\omega.$$

Thus the Yang–Mills density $e^{-S_{\mathrm{YM}}(\omega)/t}$ is $e^{-|J(\omega)|^2/2t}$.

A.7. The classical limit of μ_t

A *heuristic* calculation now shows that, for suitable $\mathcal{G}$-invariant functions F on $\mathcal{A}$, we should have

$$\lim_{t \downarrow 0} \int_{\mathcal{A}} F(\omega)\, e^{-|J(\omega)|^2/t} D\omega \sim \int_{\mathcal{A}^0/\mathcal{G}} F \,\mathrm{vol}_{\bar{\Omega}},$$

where $\mathrm{vol}_{\bar{\Omega}}$ is the volume form corresponding to the induced symplectic structure $\bar{\Omega}$ on (part of) the *moduli space of flat connections*

$$\frac{J^{-1}(0)}{\mathcal{G}} = \frac{\mathcal{A}^0}{\mathcal{G}}.$$

Here $\mathcal{A}^0$ is the set of all *flat connections*, i.e. those with curvature zero.

Combining all this leads to the conjecture that

$$\lim_{t \downarrow} \int_{\mathcal{A}_T} f(x(c_1) \cdots x(c_n))\, \mathrm{d}\nu_t^T(x) \sim \int_{\mathcal{A}^0/\mathcal{G}} f(h_u(c_1; \omega) \cdots h_u(c_n; \omega))\, \mathrm{d}\,\mathrm{vol}_{\bar{\Omega}}([\omega]), \tag{A.10}$$

where $|\omega| \in \mathcal{A}^0/\mathcal{G}$ corresponds to $\omega \in \mathcal{A}^0$, and $\sim$ indicates equality up to constant multiple.

A.8. The standard realization of $\mathcal{A}^0/\mathcal{G}$

On the surface Σ, there are loops $A_1, B_1, \ldots, A_g, B_g$ all based at o, whose homotopy classes generate the fundamental group $\pi_1(\Sigma, o)$ subject to the condition

$$\overline{B}_g \overline{A}_g B_g A_g \cdots \overline{B}_1 \overline{A}_1 B_1 A_1 = I, \tag{A.11}$$

where I is the identity element in $\pi_1(\Sigma, o)$ and equality above is in $\pi_1(\Sigma, o)$. Here, as always, g is the genus of Σ, assumed to be positive.

Assume again that G is compact, connected, simply connected, and Σ is closed, oriented of genus $g \geq 1$. Recall the product commutator map $K_g : G^{2g} \to G$ from (2). A standard result (a detailed proof of which is available in [19, Theorem 4.1] for the more general case of Yang–Mills connections on possibly non-trivial bundles) says that the mapping

$$I : \mathcal{A}^0 \to G^{2g} : \omega \mapsto (h_u(A_1; \omega) \cdots h_u(B_g; \omega)), \tag{A.12}$$

has image

$$I(\mathcal{A}^0) = K_g^{-1}(e).$$

Moreover, I induces a well-defined *bijection*

$$\bar{I} : \frac{\mathcal{A}^0}{\mathcal{G}} \to \frac{K_g^{-1}(e)}{G}, \tag{A.13}$$

where, on the right, G acts on $K_g^{-1}(e) \subset G^{2g}$ by conjugating each factor. It is this identification of the moduli space of flat connections with $K_g^{-1}(e)/G$ which we use.

It is proved in [19, Theorem 6.1] that the symplectic structure $\bar{\Omega}$ on $\mathcal{A}^0/\mathcal{G}$ induces via $\bar{I}$ the symplectic structure $\bar{\Omega}$ on $K_g^{-1}(e)^0/G$ mentioned in (9).

A.9. *The limit for curves generating* $\pi_1(\Sigma, o)$

We specialize the conjecture (A.10) to the case when $c_1, \ldots, c_n$ are the loops $A_1, \ldots, B_g$. The case of general loops $c_1, \ldots, c_n$ reduces to this special case by using the fact that $Q_t(x) \to \delta(x)$ as $t \downarrow 0$ to eliminate homotopically trivial loops. This requires some work; details are as in the proof of [18, Lemma 8.5].

Consider again the picture of our closed, oriented genus g surface Σ arising from the closed unit disk $D \subset \mathbf{R}^2 = \mathbf{C}$ by a quotient map $q : D \to \Sigma$. On ∂D mark off the points $z_k = \mathrm{e}^{2\pi \mathrm{i}k/4g}$, for $k \in \{0, 1, \ldots, 4g\}$. Let L_k denote the radial segment from the center O of D to the point z_k. Let S_k be the arc along ∂D running from z_{k-1} to z_k. The map q is injective in the interior of D and pastes S_1 with $\overline{S_3}$ (the bar indicates reverse), S_2 with $\overline{S_4}$, S_5 with $\overline{S_7}, \ldots, S_{4g-2}$ with $\overline{S_{4g}}$. Thus, for example, $\overline{q(L_0)}q(S_1)q(L_0)$ is a *loop* on the surface, which we denote as A_1. Similarly, we have the loops $B_1, A_2, B_2, \ldots, A_g, B_g$:

$$A_k \stackrel{\text{def}}{=} \overline{q(L_0)}q(S_{4k-3})q(L_0), \qquad B_k \stackrel{\text{def}}{=} \overline{q(L_0)}q(S_{4k-2})q(L_0). \tag{A.14}$$

Traversing around ∂D along the arcs S_i, and going back and forth to O along L_0, erasing segments which are traversed forwards and backwards successively, the loop $\overline{B}_g\overline{A}_g B_g A_g \cdots \overline{B}_1\overline{A}_1 B_1 A_1$ in Σ simplifies to $\overline{a(L_0)}q(\partial D)q(L_0)$. Compare with the condition (A.11).

Consider now the triangulation T' of D given by the radial segments $L_1, \ldots, L_{4g}$, and the arcs $S_1, \ldots, S_{4g}$. Unfortunately, $T = q(T')$ fails to be a triangulation of Σ because q identifies all the points z_k; but it is "nearly" a triangulation (all that is needed is a subdivision of T' using two new vertices on each arc S_k and corresponding radial segments). We will disregard this technical issue (which can be resolved with the subdivision method and the convolution technique discussed below).

Observe that $A_1, \dots, B_g$ are loops in T. The orientation of Σ is the one induced by q from the standard orientation of D. Let Δ_k be the oriented two-cells in T whose boundary is $\overline{q(L_k)}q(S_k)q(L_{k-1})$. The integral of interest to us is

$$\int_{\mathcal{A}_T} f(x(A_1), \dots, x(B_g)) \prod_{k=1}^{4g} Q_{t|\Delta_k|}(x(\partial\Delta_k))\, \mathrm{d}x, \tag{A.15}$$

where f is any continuous function on G^{2g}-invariant under the conjugation action of G and dx denotes unit-mass Haar measure as in (A.4). It will be useful to introduce a relabelling of the edge-variables $x(e_j)$ which will reflect the specific situation at hand. Let

$$x_i \stackrel{\text{def}}{=} x_{L_i}, \qquad a_k = x(q(S_{4k-3})), \qquad b_k = x(q(S_{4k-2})),$$

for $i \in \{0, 1, \dots, 4g-1\}$ and $k \in \{1, 2, \dots, g\}$. Compare with (A.14).

In the integrand in (A.15), the $f(\cdots)$ term involves x_0 but no other x_i. Integration over $x_1, \dots, x_{4g-1}$ can be carried out step-by-step using the fundamental convolution property of the heat-kernel

$$\int_G Q_r(y^{-1}z) Q_s(xy)\, \mathrm{d}y = Q_{s+r}(xz), \tag{A.16}$$

to combine all adjacent two-cells and eliminate the variables $x_1, \dots, x_{4g-1}$ from the integration. For example, x_1 appears in the integration (A.15) only as

$$\int_G Q_{t|\Delta_1|}(x_1^{-1}a_1x_0) Q_{t|\Delta_2|}(x_2^{-1}b_1x_1)\, \mathrm{d}x_1,$$

and this, by the convolution property is equal to

$$Q_{t(|\Delta_1|+|\Delta_2|)}(x_2^{-1}b_1a_1x_0).$$

Next x_2 is eliminated:

$$\int_G Q_{t|\Delta_3|}(x_3^{-1}a_1^{-1}x_2) Q_{t(|\Delta_1|+|\Delta_2|)}(x_2^{-1}b_1a_1x_0)\, \mathrm{d}x_2 = Q_{t(|\Delta_1|+|\Delta_2|+|\Delta_3|)}(x_3^{-1}a_1^{-1}b_1a_1x_0).$$

Proceeding in this way all around the circle ∂D reduces (A.15) to

$$\int f(x_0^{-1}a_1x_0, \dots, x_0^{-1}b_gx_0) Q_{t|\Sigma|}(x_0^{-1}K_g(a_1, \dots, b_g)x_0)\, \mathrm{d}x_0\, \mathrm{d}a_1 \cdots \mathrm{d}b_g.$$

Conjugation invariance of f and of the heat-kernel, combined with $\int_G \mathrm{d}x_0 = 1$, implies that x_0 drops out. Thus

$$\int_{\mathcal{A}_T} f(x(A_1) \cdots x(B_g))\, \mathrm{d}\nu_t^T(x) = \int_{G^{2g}} f(a_1, \dots, b_g) Q_{t|\Sigma|}(K_g(a_1, \dots, b_g))\, \mathrm{d}a_1 \cdots \mathrm{d}b_g, \tag{A.17}$$

where $|\Sigma|$ is the area of Σ, obtained as the sum of the areas $|\Delta_i|$.

Specializing our conjecture (A.10) to this situation now gives Theorem 1 (without the constant of proportionality).

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